

# SOLENOIDAL MINIMAL SETS FOR FOLIATIONS

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## 1. INTRODUCTION

We recall the most general definition of an  $n$ -dimensional solenoid:

**DEFINITION 1.1** ([21, 25, 8]). *An  $n$ -dimensional solenoid is an inverse limit space*

$$\mathcal{S} = \varprojlim \{p_{\ell+1}: L_{\ell+1} \rightarrow L_\ell\}$$

where for  $\ell \geq 0$ ,  $L_\ell$  is a closed, oriented,  $n$ -dimensional manifold, and the maps  $p_{\ell+1}: L_{\ell+1} \rightarrow L_\ell$  are smooth, orientation-preserving proper covering maps.

The limit topological space  $\mathcal{S}$  is a lamination [7], with foliation  $\mathcal{F}_\mathcal{S}$  having leaves of dimension  $n$ . The initial manifold  $L_0$  is called the *base* of the solenoid  $\mathcal{S}$ , and each covering  $p_{\ell+1}: L_{\ell+1} \rightarrow L_\ell$  is called a *bonding map*. It is not assumed that the bonding maps are normal coverings, only that the images  $p_{\ell+1}: \pi_1(L_{\ell+1}, x_{\ell+1}) \rightarrow \pi_1(L_\ell, x_\ell)$  have finite index for each  $\ell \geq 0$ , where  $x_0 \in L_0$  is a basepoint and  $x_\ell \in L_\ell$  is chosen so that  $p_{\ell+1}(x_{\ell+1}) = x_\ell$ .

There is a natural fibration map  $p_*: \mathcal{S} \rightarrow L_0$ , such that the restriction of  $p_*$  to each leaf  $L \subset \mathcal{S}$  of  $\mathcal{F}_\mathcal{S}$  is a covering of  $L_0$ . Given a Riemannian metric on the base  $L_0$ , the covering maps can be used to define Riemannian metrics on each  $L_\ell$ . Then each leaf  $L$  of  $\mathcal{F}_\mathcal{S}$  inherits a Riemannian metric such that the covering map  $p_*: L \rightarrow L_0$  is a local isometry. Let

$$(1) \quad \hat{p}_\ell = p_1 \circ \cdots \circ p_\ell: L_\ell \longrightarrow L_0$$

Set  $\Gamma = \pi_1(L_0, x_0)$  and  $\Gamma_\ell = \text{image}\{(\hat{p}_\ell)_*: \pi_1(L_\ell, x_\ell) \rightarrow \Gamma\}$ , then let  $\mathfrak{K}_\ell = \Gamma/\Gamma_\ell$  be the quotient set, which is a group if all bonding maps are normal coverings. The fiber  $\mathbb{K}_* \equiv p_*^{-1}(x_0)$  of  $p_*: \mathcal{S} \rightarrow L_0$  is then identified with the inverse limit

$$(2) \quad \mathbb{K}_* \equiv p_*^{-1}(x_0) \cong \varprojlim \{\mathfrak{K}_{\ell+1} \longrightarrow \mathfrak{K}_\ell\}$$

which is compact, totally-disconnected, and if all bonding maps are normal, a topological group.

The case where each  $L_\ell = \mathbb{S}^1$  and the bonding maps  $p_\ell$  are orientation-preserving covering maps with degree  $d_\ell > 1$  is the classic example of a solenoid. The limit space  $\mathcal{S}$  then has a minimal flow, which covers the usual rotation flow of  $\mathbb{S}^1$ , and the fiber  $\mathbb{K}_*$  is homeomorphic to a standard  $\vec{d}$ -adic Cantor set, for  $\vec{d} = (d_1, d_2, \dots)$ . The study minimal sets homeomorphic to a solenoid has an extensive history in topological dynamics.

The first author, in the paper [8] with Robbert Fokkink, asked when a solenoid with leaves of dimension greater than one, can be embedded into a manifold. Let  $\mathbb{B}^q \subset \mathbb{R}^q$  denote the open unit disk, then they consider the product space  $L_0 \times \mathbb{B}^q$  for  $q > 1$ , and ask for conditions that imply  $\mathcal{S}$  embeds into  $L_0 \times \mathbb{B}^q$  so that the bundle projection induces the limit map  $p_*$ . Note that the existence of such an embedding implies an embedding of the fiber Cantor set  $\mathbb{K}_* \subset \mathbb{B}^q$ . For the case where  $q = 2$ , they give a complete solution to the problem.

In this same paper [8], the authors also ask, for  $n > 1$ , when can an  $n$ -dimensional solenoid be realized as a minimal set for a  $C^r$ -foliation  $\mathcal{F}$  on  $L_0 \times \mathbb{B}^q$ , for some  $r \geq 0$ ? It is assumed that

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the leaves of  $\mathcal{F}$  are all coverings of the base  $L_0$ . In this case, not only is the fiber Cantor set  $\mathbb{K}_*$  embedded in  $\mathbb{B}^q$ , but the monodromy automorphisms of the fiber of  $p_*: \mathcal{S} \rightarrow L_0$  are realized via  $C^r$ -holonomy maps of the foliation  $\mathcal{F}$ , which is a very strong condition. It seems that very little is known about the answer to this question, especially when the differentiability  $r \geq 1$ .

The realization problem for solenoids as minimal sets is a special case of a broader problem in the study of foliations on closed manifolds.

**PROBLEM 1.2.** *Given a lamination  $X$  with foliation  $\mathcal{F}_X$ , and  $r \geq 0$ , when does there exist a  $C^r$ -foliation  $\mathcal{F}$  of a closed manifold  $M$ , such that  $\mathcal{F}$  has a minimal set homeomorphic as foliated spaces with  $(X, \mathcal{F}_X)$ ?*

The purpose of this note is to give a construction of higher-dimension solenoids which naturally arise as minimal sets for  $C^r$ -foliations, for any  $r \geq 0$ . Our approach is related to the proof of Theorem 14 in [8], but is equally inspired by the ideas of stability theory for  $C^1$ -foliations [19, 27, 28, 33].

In fact, our construction begins with an arbitrary  $C^r$ -foliation  $\mathcal{F}_0$  which contains a compact leaf  $L_0$  satisfying certain “instability conditions”. We construct a family  $\{\mathcal{F}_\ell \mid \ell = 1, 2, \dots\}$  of  $C^r$ -perturbations of  $\mathcal{F}_0$ , whose limit is a foliation  $\mathcal{F}$  which contains a solenoid minimal set. For each  $\ell \geq 0$ , the foliation  $\mathcal{F}_\ell$  contains the approximating tower for  $\mathcal{S}$  up to height  $\ell$ . That is, via a perturbation, the compact leaf  $L_0$  “blows-up” into a solenoid obtained as an inverse limit of finite coverings of  $L_0$ . The degree of differentiability of the limit foliation  $\mathcal{F}$  depends upon the choices of the maps in the inverse limit construction, and can be made  $C^r$  with suitable such choices. Let  $\mathbb{D}^q$  denote the closed unit disk in  $\mathbb{R}^q$ .

**THEOREM 1.3.** *Let  $L_0$  be a closed oriented manifold of dimension  $n$ , with  $H^1(L_0; \mathbb{R}) \neq 0$ . Let  $q \geq 2$ ,  $r \geq 1$ , and  $\mathcal{F}_0$  denote the product foliation of  $M = L_0 \times \mathbb{D}^q$ . Then there exists a  $C^r$ -foliation  $\mathcal{F}$  of  $M$  which is  $C^r$ -close to  $\mathcal{F}_0$ , such that  $\mathcal{F}$  is a volume-preserving, distal foliation, and satisfies*

- (1)  $L_0$  is a leaf of  $\mathcal{F}$
- (2)  $\mathcal{F} = \mathcal{F}_0$  near the boundary of  $M$
- (3)  $\mathcal{F}$  has a minimal set  $\mathcal{S}$  which is a generalized solenoid with base  $L_0$
- (4) each leaf  $L \subset \mathcal{S}$  is a covering of  $L_0$ .

The foliated manifold  $(M, \mathcal{F})$  functions as a “foliated plug”, in the spirit of the constructions of Wilson [34], Schweitzer [26] and Kuperberg [18]. Thus, we can use it to insert a solenoidal minimal set, for a given foliation  $\mathcal{F}$ , which has an open neighborhood of a compact leaf with product foliation.

**COROLLARY 1.4.** *Let  $\mathcal{F}_0$  be a  $C^r$ -foliation of codimension  $q \geq 2$  on a manifold  $M$ . Let  $L_0$  be a compact leaf with  $H^1(L_0; \mathbb{R}) \neq 0$ , and suppose that  $\mathcal{F}_0$  is a product foliation in some open neighborhood  $U$  of  $L_0$ . Then there exists a foliation  $\mathcal{F}$  on  $M$  which is  $C^r$ -close to  $\mathcal{F}_0$ , and  $\mathcal{F}$  has a solenoidal minimal set contained in  $U$  with base  $L_0$ . If  $\mathcal{F}_0$  is a distal foliation, then  $\mathcal{F}$  is also distal.*

Note that Corollary 1.4 is a type of “anti-stability” result. Recall that Theorem 2 of Langevin and Rosenberg [19] states that if  $H^1(L_0; \mathbb{R}) = \{0\}$  and  $L_0$  has a product open neighborhood, and if  $\mathcal{F}'$  is a sufficiently close  $C^1$ -perturbation of  $\mathcal{F}$ , then there is a compact leaf  $L'_0$  of  $\mathcal{F}'$  near to  $L_0$  which has a product open neighborhood, hence  $\mathcal{F}'$  has no solenoidal minimal set near to  $L'_0$ .

The minimal sets of an equicontinuous foliation are submanifolds [23, 1], hence if  $\mathcal{F}$  has a solenoidal minimal set, then  $\mathcal{F}$  cannot be equicontinuous. The conclusion that  $\mathcal{F}$  is distal in Corollary 1.4 also implies that it does not have any leafwise holonomy which is partially hyperbolic, which implies the minimal set  $\mathcal{S}$  is parabolic in terms of the classification scheme of [13].

There are many articles in the literature giving constructions of solenoidal minimal sets for flows, or equivalently, of  $C^1$ -diffeomorphisms with invariant Cantor sets on which the action is minimal and quasi-periodic (see, for example, the works [4, 10, 9, 15, 20, 30].) The idea of the constructions in all cases is similar, invoking cascades of periodic orbits embedded in invariant solid tori, and a deformation of the action between the nested tori (or invariant circles in the case of diffeomorphisms). The key observation of this paper is to regard these invariant solid tori as foliated flat disk-bundles,

and the deformations of the action as deformations of the flat bundle structures. Thus formulated, it becomes clear how to extend the constructions to flat bundles from base manifolds which are circles, to higher dimensional tori. There are a variety of technical, “bookkeeping” issues to keep track of, including the fairly obscure issue of when the choices in the construction yield a  $C^r$ -foliation for  $r > 1$ . However, for some applications of these examples (see [14]), it is critical that the examples are at least  $C^2$ .

The construction of the foliation  $\mathcal{F}$  on  $M = L_0 \times \mathbb{D}^q$  given in section 3 uses a geometric approach via embedded tori, and highlights the foliation aspect of the construction. However, to prove the resulting foliation is  $C^r$ , we reformulate the construction in section 4 in terms of the global holonomy maps  $h: \mathbb{Z}^k \rightarrow \text{Diff}^r(\mathbb{D}^q)$ . This formulation also makes the comparison of the properties of our examples with those of the classical examples of actions on the disk more transparent.

**THEOREM 1.5.** *Let  $q = 2m$  or  $q = 2m + 1$ ,  $r \geq 1$  and  $\delta > 0$ . Let  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$  be a sequence of representations which tend to zero in norm sufficiently rapidly. Then there exists a  $C^r$ -action  $h: \mathbb{Z}^k \rightarrow \text{Diff}^r(\mathbb{D}^q)$  such that for each generator  $\vec{e}_j$  of  $\mathbb{Z}^k$ , the map  $h_j = h(\vec{e}_j)$  is  $\delta$ -close to the identity map in the uniform  $C^r$ -topology, and satisfies:*

- (1)  $\vec{0} \in \mathbb{D}^q$  is a fixed-point for  $h_j$ ;
- (2)  $h_j$  is the identity map in an open neighborhood of the boundary  $\mathbb{S}^{q-1}$ ;
- (3)  $h_j$  preserves the standard volume form on  $\mathbb{D}^q$ ;
- (4)  $h_j$  has zero topological entropy.

Moreover, the group action  $h$  satisfies:

- (5) there exists an  $h$ -invariant Cantor set  $\widehat{K} \subset \mathbb{B}^q$  on which the group action is minimal;
- (6) the group action  $h$  on  $\mathbb{D}^q$  is distal;
- (7) there exists a nested sequence of  $h$ -invariant open sets  $U_\ell \subset \mathbb{D}^q$  which converge to  $\widehat{K}$ , and such that the restriction of the action  $h$  to  $U_\ell$  is periodic, with the restricted action on  $U_\ell$  induced from the finite representation  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m \rightarrow \mathbb{Q}^m/\mathbb{Z}^m$ .

It is worth noting that property (1.5.7) of Theorem 1.5 implies that the collection of Borel equivalence relations defined by the actions constructed in this paper are uncountable in number, and are essentially unclassifiable in the sense of descriptive set theory [17, 31, 32, 12].

Also note, that these higher dimensional solenoids typically do not decompose into a product of one-dimensional solenoids. This follows from Appendix I of Pontrjagin [24].

Section 5 illustrates the method of the construction of the action  $h$  in the classical case where  $k = 1$  and  $q = 2$ ; that is, for a single diffeomorphism of the 2-disk, as this helps clarify the role of the choices made with the degree of differentiability obtained for the resulting actions. The final section 6 discusses additional results and open questions.

## 2. FLAT BUNDLES

We recall some of the basic facts about flat vector bundles, and then construct examples which play a decisive role in the proof of Theorem 1.3.

Choose a basepoint  $x_0 \in L_0$ , and set  $\Gamma = \pi_1(L_0, x_0)$ . Let  $\Gamma$  act on the right as deck transformations of the universal cover  $\widetilde{L}_0 \rightarrow L_0$ .

Let  $\rho: \Gamma \rightarrow \mathbf{SO}(q)$  be an orthogonal representation. Then  $\Gamma$  acts on the left as isometries of  $\mathbb{R}^q$ ,  $\vec{v} \mapsto \rho(\gamma)(\vec{v})$ . Define a flat  $\mathbb{R}^q$ -bundle by

$$(3) \quad \mathbb{E}_\rho^q = (\widetilde{L}_0 \times \mathbb{R}^q) / (\widetilde{y} \cdot \gamma, \vec{v}) \sim (\widetilde{y}, \rho(\gamma)(\vec{v})) \longrightarrow L_0$$

For a closed path  $\sigma_\gamma: [0, 1] \rightarrow L_0$  with  $\sigma_\gamma(0) = \sigma_\gamma(1) = x_0$  which represents the homotopy class  $\gamma$ , the holonomy of the bundle  $\mathbb{E}_\rho$  along the path  $\sigma_\gamma$  is  $\rho(\gamma)$ , as seen via the identification in (3).

The defining property of a flat bundle  $\mathbb{E} \rightarrow L_0$  is that its horizontal distribution is integrable, and hence defines a foliation of  $\mathbb{E}$  by the integral submanifolds of the horizontal distribution. In the case where the flat bundle  $\mathbb{E}$  is given explicitly via a group action, as for  $\mathbb{E}_\rho^q$ , the leaves of the foliation  $\mathcal{F}_\rho$  have an explicit description. For each  $\vec{v} \in \mathbb{R}^q$ , let  $\tilde{L}_{\vec{v}} = \tilde{L}_0 \times \{\vec{v}\}$  be the leaf through  $\vec{v}$  of the product foliation on  $(\tilde{L}_0 \times \mathbb{R}^q)$ . The product foliation is  $\Gamma$ -equivariant, so descends to a foliation denoted by  $\mathcal{F}_\rho$ , and the leaf  $\tilde{L}_{\vec{v}}$  descends to a leaf denoted by  $L_{\vec{v}} \subset \mathbb{E}_\rho^q$ .

The action of  $\Gamma$  on  $\mathbb{R}^q$  via  $\rho$  preserves the usual norm,  $\|\vec{x}\|^2 = x_1^2 + \cdots + x_q^2$ , so for all  $\epsilon > 0$  restricts to actions on the subsets

$$\mathbb{B}_\epsilon^q = \{\vec{x} \mid \|\vec{x}\| < \epsilon\}, \quad \mathbb{D}_\epsilon^q = \{\vec{x} \mid \|\vec{x}\| \leq \epsilon\}, \quad \mathbb{S}_\epsilon^{q-1} = \{\vec{x} \mid \|\vec{x}\| = \epsilon\}$$

Let  $\mathbb{B}_{\epsilon,\rho}^q \rightarrow L_0$  (respectively,  $\mathbb{D}_{\epsilon,\rho}^q$  or  $\mathbb{S}_{\epsilon,\rho}^{q-1}$ ) denote the  $\mathbb{B}_\epsilon^q$ -subbundle (respectively,  $\mathbb{D}_\epsilon^q$  or  $\mathbb{S}_\epsilon^{q-1}$ ) of  $\mathbb{E}_\rho^q \rightarrow L_0$ . The foliation  $\mathcal{F}_\rho$  then restricts to a foliation on each of these subbundles.

The most familiar example is for  $L_0 = \mathbb{S}^1$ , where  $\Gamma = \pi_1(\mathbb{S}^1, x) = \mathbb{Z}$ . Given any  $\alpha \in \mathbb{R}$ , define the representation  $\rho: \mathbb{Z} \rightarrow \mathbf{SO}(2)$  by  $\rho(n) = \exp(2\pi\sqrt{-1}\alpha \cdot n)$ . Then  $\mathbb{E}_\rho^2$  is the flat vector bundle over  $\mathbb{S}^1$ , such that the holonomy along the base  $\mathbb{S}^1$  rotates the fiber by  $\rho(1) = \exp(2\pi\sqrt{-1}\alpha)$ .

In general, the bundle  $\mathbb{E}_\rho^q \rightarrow L_0$  need not be a product vector bundle, and may even have non-trivial Euler class when  $q$  is even [16, 22, 29, 35].

**PROPOSITION 2.1.** *Let  $\rho_t: \Gamma \rightarrow \mathbf{SO}(q)$ , for  $0 \leq t \leq 1$ , be a smooth 1-parameter family of representations such that  $\rho_0$  is the trivial map, and  $\rho_1 = \rho$ . Then  $\rho_t$  canonically defines an isomorphism of vector bundles  $\mathbb{E}_\rho^q \cong L \times \mathbb{R}^q$ .*

**Proof:** The family of representations defines a family of flat bundles  $\mathbb{E}_{\rho_t}^q$  over the product space  $L \times [0, 1]$ . The coordinate vector field  $\partial/\partial t$  along  $[0, 1]$  lifts to a vector field  $\tilde{v}$  on the product,  $\tilde{\mathbb{E}}^q = (\tilde{L}_0 \times [0, 1]) \times \mathbb{R}^q$ . The group  $\Gamma$  acts on  $\tilde{\mathbb{E}}^q$  via the action of  $\rho_t$  on each slice  $\tilde{\mathbb{E}}_t^q = (\tilde{L}_0 \times \{t\}) \times \mathbb{R}^q$ , and the vector field  $\tilde{v}$  is clearly  $\Gamma$  invariant, as  $\rho_t$  acts on the vector space  $\mathbb{R}^q$  but fixes the tangent bundle to  $[0, 1]$ . Thus,  $\tilde{v}$  descends to a vector field  $\vec{v}$  on  $\tilde{\mathbb{E}}^q/\Gamma$ . The flow of  $\vec{v}$  on  $\tilde{\mathbb{E}}^q$  preserves the fibers, so the flow of  $\vec{v}$  acts via bundle isomorphisms on  $\tilde{\mathbb{E}}^q/\Gamma \rightarrow L_0 \times [0, 1]$ .

The time-one flow of  $\vec{v}$  defines an isotopy between the bundles  $\mathbb{E}_{\rho_0}^q$  and  $\mathbb{E}_{\rho_1}^q$ , which induces a bundle isomorphism between them. The initial bundle  $\mathbb{E}_{\rho_0}^q$  is a product, hence the same holds for  $\mathbb{E}_{\rho_1}^q$ .  $\square$

The key point is that the bundle isomorphism between  $\mathbb{E}_{\rho_0}^q$  and  $\mathbb{E}_{\rho_1}^q$  is canonical, and in particular, depends smoothly on the path  $\rho_t$ .

In general, a representation  $\rho$  need not be homotopic to the trivial representation, so this hypothesis is very strong. However, when  $\rho: \Gamma \rightarrow \mathbf{SO}(q)$  factors through either a free abelian group  $\mathbb{Z}^k$ , or free non-abelian group  $\mathbb{F}^k$ , then such an isotopy always exists. For example, if  $\rho = \alpha \circ \pi: \Gamma \rightarrow \mathbb{Z}^k \rightarrow \mathbf{SO}(q)$ , then  $\rho$  always satisfies this condition; our construction makes use of the case where  $\rho$  factors through  $\mathbb{Z}^k$ . We consider the special case of representations  $\rho: \mathbb{Z}^k \rightarrow \mathbf{SO}(q)$  in more detail.

Let  $m$  be the greatest integer such that  $2m \leq q$ . For  $q = 2m$ , for convenience of notation, we identify  $\mathbb{R}^q \cong \mathbb{C}^m$ , via the map

$$\vec{x} = (x_1, x_2, \dots, x_{2m-1}, x_{2m}) \mapsto \vec{z} = (z_1, \dots, z_m)$$

where  $z_i = x_{2i-1} + \sqrt{-1} \cdot x_{2i}$ . For  $q = 2m + 1$ , identify  $\mathbb{R}^q = \mathbb{R}^{2m} \times \mathbb{R} \cong \mathbb{C}^m \times \mathbb{R}$  in the same fashion. The  $m$ -torus  $\mathbb{T}^m$  will be written as

$$\mathbb{T}^m = \{\vec{w} = (w_1, \dots, w_m) \mid w_i \in \mathbb{C}, |w_i| = 1, \text{ for all } 1 \leq i \leq m\}$$

Identify  $\mathbb{T}^m$  with a maximal torus in  $\mathbf{SO}(q)$  via its action by coordinate multiplication, for  $\vec{z} \in \mathbb{C}^m$ ,

$$\begin{aligned} (4) \quad \vec{w} \cdot \vec{z} &= (w_1 \cdot z_1, \dots, w_m \cdot z_m), \quad q = 2m \\ \vec{w} \cdot (\vec{z}, x_{2m+1}) &= (w_1 \cdot z_1, \dots, w_m \cdot z_m, x_{2m+1}), \quad q = 2m + 1 \end{aligned}$$

Given  $\vec{a} = (a_1, \dots, a_m) \in \mathbb{R}^m$ , set

$$(5) \quad \overline{\exp}(\vec{a}) = (\exp(2\pi\sqrt{-1} \cdot a_1), \dots, \exp(2\pi\sqrt{-1} \cdot a_m)) \in \mathbb{T}^m$$

Let  $\alpha: \Gamma \rightarrow \mathbb{R}^m$  be a representation. Define a parametrized family of representations, for  $0 \leq t \leq 1$ ,

$$(6) \quad \rho_t^\alpha: \Gamma \rightarrow \mathbb{T}^m \subset \mathbf{SO}(q), \quad \rho_t^\alpha(\gamma)(\vec{v}) = \overline{\exp}(t \cdot \alpha(\gamma)) \cdot \vec{v}$$

Then we have:

**PROPOSITION 2.2.** *Each representation  $\alpha: \Gamma \rightarrow \mathbb{R}^m$  defines a foliation  $\mathcal{F}_\alpha$  of  $L_0 \times \mathbb{S}_\epsilon^{q-1}$  whose leaves cover  $L_0$ . Moreover, if the image of  $\alpha$  is contained in the rational points  $\mathbb{Q}^m \subset \mathbb{R}^m$ , then all leaves of  $\mathcal{F}_\alpha$  are compact.*

**Proof:** Given  $\alpha: \Gamma \rightarrow \mathbb{R}^m$ , then the family  $\rho_t^\alpha$  is an isotopy from  $\rho = \rho_1^\alpha$  to the trivial representation, so  $\mathbb{S}_{\epsilon, \rho}^{q-1} \rightarrow L_0$  is bundle-isomorphic to the product bundle  $L_0 \times \mathbb{S}^{q-1}$ . If the image of  $\alpha$  is contained in  $\mathbb{Q}^m$ , then the image of  $\rho^\alpha$  is a finite subgroup of  $\mathbf{SO}(q)$ , hence each leaf of  $\mathcal{F}_\rho$  is a finite covering of  $L_0$  hence is compact. Let  $\mathcal{F}_\alpha$  denote the foliation which is the image of  $\mathcal{F}_\rho$  under the fiber-preserving diffeomorphism  $\mathbb{S}_{\epsilon, \rho}^{q-1} \cong L_0 \times \mathbb{S}_\epsilon^{q-1}$  induced by the isotopy  $\rho_t^\alpha$ .  $\square$

In the case where  $q = 2$ ,  $\mathbf{SO}(q) \cong \mathbb{S}^1$  and  $L_0 = \mathbb{S}^1$ , then  $\alpha: \mathbb{Z} \rightarrow \mathbb{R}$  is determined by the real number  $\alpha = \alpha(1)$ , and  $\mathcal{F}_\alpha$  is the foliation of the 2-torus  $\mathbb{T}^2$  by lines of “slope”  $\alpha$ . Note that for the abstract flat bundle  $\mathbb{E}_\rho^2$  the holonomy is rotation of the fiber  $\mathbb{S}^1$  by the congruence class of  $\alpha$  modulo  $\mathbb{Z}$ . However, using the homotopy  $\rho_t^\alpha$  we are able to define the slope of the leaves  $\mathcal{F}_\rho$  using the explicit product structure. A similar phenomenon holds for the general case of  $\rho$  defined by  $\alpha: \Gamma \rightarrow \mathbb{R}^m$  although it is not as immediate to imagine the foliation  $\mathcal{F}_\alpha$  on the product space  $L_0 \times \mathbb{S}_\epsilon^{q-1}$ . However, there is a standard observation which obviates this problem.

For a vector  $\vec{x} \in \mathbb{R}^k$  let  $[\vec{x}] \in \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$  denote the coset of  $\vec{x} \in \mathbb{R}^k$ . Let  $[\vec{0}]$  denote the basepoint defined by the origin in  $\mathbb{R}^k$ . The fundamental group  $\pi_1(\mathbb{T}^k, [\vec{0}])$  acts on the universal cover  $\mathbb{R}^k$  via translations, so is canonically identified with the integer lattice  $\mathbb{Z}^k$ .

**PROPOSITION 2.3.** *Suppose that  $H^1(L_0, \mathbb{R}) \cong \mathbb{R}^k$  for  $k \geq 1$ . Then there is a surjective map  $\pi: \Gamma = \pi_1(L_0, x_0) \rightarrow \mathbb{Z}^k$ . Moreover, there is a smooth map  $\tau: L_0 \rightarrow \mathbb{T}^k$  with  $\tau(x_0) = [\vec{0}] \in \mathbb{T}^k$ , and  $\pi = \tau_*: \pi_1(L_0, x_0) \rightarrow \pi_1(\mathbb{T}^k, [\vec{0}])$ .*  $\square$

**PROPOSITION 2.4.** *Given a foliation  $\hat{\mathcal{F}}$  of  $\mathbb{T}^k \times \mathbb{D}^q$  which is transverse to the factor  $\mathbb{D}^q$ , then  $\tau: L_0 \rightarrow \mathbb{T}^k$  induces a foliation  $\mathcal{F}$  on  $L_0 \times \mathbb{D}^q$  whose global holonomy map is the composition of  $\tau_*$  with the holonomy map  $h_{\hat{\mathcal{F}}}: \mathbb{Z}^k \rightarrow \text{Diff}(\mathbb{D}^q)$ .*  $\square$

In the following, we construct foliations of  $N_0 = \mathbb{T}^k \times \mathbb{D}^q$ , then obtain foliations of  $M = L_0 \times \mathbb{D}^q$  by Propositions 2.3 and 2.4.

Given  $\alpha = (\alpha_1, \dots, \alpha_m): \mathbb{Z}^k \rightarrow \mathbb{R}^m$ , there is a unique linear extension to  $\mathbb{R}^k$ , also denoted by  $\alpha: \mathbb{R}^k \rightarrow \mathbb{R}^m$ . The extension  $\alpha$  is used to define a diagonal action

$$(7) \quad \begin{aligned} T_\alpha: \mathbb{R}^k \times (\mathbb{T}^k \times \mathbb{T}^m) &\rightarrow (\mathbb{T}^k \times \mathbb{T}^m) \\ T_\alpha(\vec{\xi})([\vec{x}], [\vec{y}]) &= ([\vec{x} + \vec{\xi}], [\vec{y} + \alpha(\vec{\xi})]) \end{aligned}$$

for  $\vec{\xi} \in \mathbb{R}^k$ . The orbits of  $T_\alpha$  define a foliation  $\mathcal{F}_\alpha$  on  $\mathbb{T}^k \times \mathbb{T}^m$  where the leaf through  $(x, y) = ([\vec{x}], [\vec{y}]) \in \mathbb{T}^k \times \mathbb{T}^m$  is

$$(8) \quad L_{(x, y)} = \{([\vec{x} + \vec{\xi}], [\vec{y} + \alpha(\vec{\xi})]) \mid \vec{\xi} \in \mathbb{R}^k\}$$

This is just the standard “linear foliation” of  $\mathbb{T}^{k+m}$  by  $k$ -planes, whose “slope” is defined by the map  $\alpha$ . If we have a smooth family of representations,  $\alpha_t: \mathbb{Z}^k \rightarrow \mathbb{R}^m$ ,  $0 \leq t \leq 1$ , then we obtain a smooth family of foliations  $\mathcal{L}_{\alpha_t}$  on  $\mathbb{T}^k \times \mathbb{T}^m$ .

The linear representation  $\alpha: \mathbb{R}^k \rightarrow \mathbb{R}^m$  can be composed with the action of  $\mathbb{T}^m$  on  $\mathbf{SO}(q)$  as in (4) to obtain an isometric action  $\rho^\alpha$  of  $\mathbb{R}^k$  on  $\mathbb{R}^q$ , which restricts to actions on each of the subsets  $\mathbb{D}_\epsilon^q$ ,

$\mathbb{B}_\epsilon^q$  and  $\mathbb{S}_\epsilon^{q-1}$  of  $\mathbb{R}^q$ . We thus obtain corresponding diagonal actions on the corresponding product spaces. The product actions are also denoted by  $T_\alpha$ , where for example, we have

$$(9) \quad T_\alpha: \mathbb{R}^k \times (\mathbb{T}^k \times \mathbb{D}_\epsilon^q) \rightarrow (\mathbb{T}^k \times \mathbb{D}_\epsilon^q)$$

The orbits of  $T_\alpha$  define a foliation of  $\mathbb{T}^k \times \mathbb{D}_\epsilon^q$ , again denoted by  $\mathcal{F}_\alpha$ . Note that the action of  $T_\alpha$  preserves the spherical submanifolds  $\mathbb{T}^k \times \mathbb{S}_{\epsilon'}^{q-1}$  for  $0 \leq \epsilon' \leq \epsilon$ .

### 3. CONSTRUCTING THE PLUG

In this section, we construct a  $C^0$ -foliation  $\mathcal{F}$  with solenoidal minimal set on  $L_0 \times \mathbb{D}^q$ . The idea is to first construct a foliation  $\widehat{\mathcal{F}}$  of  $\mathbb{T}^k \times \mathbb{D}^q$  as the limit of a sequence of foliations  $\widehat{\mathcal{F}}_\ell$  defined inductively, then use Proposition 2.4 to pull the foliation back to  $L_0 \times \mathbb{D}^q$ . The induction starts with the product foliation  $\widehat{\mathcal{F}}_0$  on  $\mathbb{T}^k \times \mathbb{D}^q$ . The construction of  $\widehat{\mathcal{F}}_1$  from  $\widehat{\mathcal{F}}_0$  is technically simpler than that of  $\widehat{\mathcal{F}}_{\ell+1}$  from  $\widehat{\mathcal{F}}_\ell$  for  $\ell > 0$ ; the construction of  $\widehat{\mathcal{F}}_{\ell+1}$  from  $\widehat{\mathcal{F}}_\ell$  involves the complete inductive procedure. There are many choices involved in the construction, which influence the topological and differentiable type of the solenoidal minimal set of the resulting foliation  $\widehat{\mathcal{F}}$ .

Before beginning the inductive construction, we establish some further notations and choices.

Let  $\vec{e}_j = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^k$ , where the sole non-zero entry is in the  $j^{\text{th}}$ -coordinate. The set  $\{\vec{e}_1, \dots, \vec{e}_k\}$  forms the *standard basis* for  $\mathbb{Z}^k$ .

For each  $\ell \geq 1$ , choose a representation  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$ , and set  $\alpha_\ell(\vec{e}_j) = \vec{a}_{\ell,j} \in \mathbb{Q}^m$ . Let  $\alpha_0$  denote the trivial representation, with  $\alpha_0(\vec{e}_j) = \vec{0}$  for all  $1 \leq j \leq k$ .

Let  $\Delta_\ell \subset \mathbb{Q}^m$  denote the image of  $\alpha_\ell$ . Given  $\delta > 0$ , we say that  $\alpha_\ell$  is  $\delta_\ell$ -thin if the generators satisfy  $0 < \|\vec{a}_{\ell,j}\| < \delta_\ell$  for all  $1 \leq j \leq k$ . Here, we use the standard inner product norm on  $\mathbb{R}^m$ . In this section, we assume only that the sequence of positive numbers  $\{\delta_\ell \mid i = 1, 2, \dots\}$  satisfy  $\delta_\ell \rightarrow 0$ , which is sufficient to guarantee that  $\widehat{\mathcal{F}}_\ell$  converges to a  $C^0$ -foliation  $\widehat{\mathcal{F}}$  of  $\mathbb{T}^k \times \mathbb{D}^q$ .

The representations  $\rho_\ell = \rho^{\alpha_\ell}: \mathbb{Z}^k \rightarrow \mathbb{T}^m \subset \mathbf{SO}(q)$  are obtained as in (6), so that for  $\vec{v} \in \mathbb{R}^q$ ,

$$\rho_\ell(\gamma)(\vec{v}) = \overline{\text{exp}}(\alpha_\ell(\gamma)) \cdot \vec{v}$$

The image of  $\rho_\ell$  is isomorphic to the finite subgroup  $\mathfrak{G}_\ell = \Delta_\ell / \mathbb{Z}^m \subset \mathbb{Q}^m / \mathbb{Z}^m$ . The image of  $\rho_0$  is the identity, while we assume that all other subgroups  $\mathfrak{G}_\ell$  for  $i \geq 1$  are non-trivial. This will be the case if  $\alpha_\ell$  is  $\delta_\ell$ -thin for  $\delta_\ell < 1$ , for all  $\ell \geq 1$ .

Let  $\Lambda_\ell = \ker\{\rho_\ell: \mathbb{Z}^k \rightarrow \mathbf{SO}(q)\} \subset \mathbb{Z}^k$ . Then  $\Lambda_0 = \mathbb{Z}^k$ , and  $\mathfrak{G}_\ell \cong \mathbb{Z}^k / \Lambda_\ell$ . Note that there are covering maps  $p_\ell: \mathbb{R}^k / \Lambda_\ell \rightarrow \mathbb{R}^k / \mathbb{Z}^k$ ; these are the bonding maps of the solenoid we construct.

For each  $\ell \geq 1$ , choose a set of generators  $\{e_1^\ell, \dots, e_k^\ell\}$  with positive orientation for  $\subset \Lambda_\ell$ , which determine an isomorphism  $\phi_\ell: \mathbb{Z}^k \rightarrow \Lambda_\ell$  given by  $\phi_\ell(c_1, \dots, c_k) = \sum_{j=1}^k c_j e_j^\ell$ . Note that two such sets of generators are related by a matrix  $A \in \mathbf{SL}(k, \mathbb{Z})$ , and the corresponding maps are related by  $\phi'_\ell = \phi_\ell \circ A$ . Let  $\phi_\ell: \mathbb{R}^k \rightarrow \mathbb{R}^k$  also denote the extension of  $\phi_\ell$  to a linear isomorphism of  $\mathbb{R}^k$ , and  $\overline{\phi}_\ell: \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k \rightarrow \mathbb{R}^k / \Lambda_\ell$  denote the induced map on quotients.

**DEFINITION 3.1.**  $\Phi_\ell: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$  is defined inductively: set  $\Phi_1 = \phi_1$ ; and for  $\ell > 1$ , set  $\Phi_\ell = \Phi_{\ell-1} \circ \phi_\ell$ . Let  $\Gamma_\ell = \text{Im}(\Phi_\ell) \subset \mathbb{Z}^k = \Gamma_0$ .

The groups  $\Gamma_\ell$  correspond to the groups introduced after Definition 1.1.

Choose a non-increasing smooth function  $\mu: [0, 1] \rightarrow [0, 1]$  such that

$$\mu(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 2/3 \\ 0 & \text{if } 3/4 \leq s \leq 1 \end{cases}$$

For each  $\ell \geq 1$ , we extend the representation  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$  to a continuous family  $\alpha_{\ell,t}: \mathbb{Z}^k \rightarrow \mathbb{R}^m$ ,  $0 \leq t \leq 1$ , by setting  $\alpha_{\ell,t}(\vec{e}_j) = \mu(t) \cdot \vec{a}_{\ell,j}$  for  $1 \leq j \leq k$ . Correspondingly,  $\alpha_{\ell,t}$  determines a smooth

family of representations,  $\rho_{\ell,t}: \mathbb{Z}^k \rightarrow \mathbb{T}^m$  where

$$(10) \quad \rho_{\ell,j,t}(\vec{v}) = \rho_{\ell,t}(\vec{e}_j)(\vec{v}) = \overline{\text{exp}}(\mu(t) \cdot \vec{a}_{\ell,j}) \cdot \vec{v}$$

Note that for  $3/4 \leq t \leq 1$ ,  $\rho_{\ell,t}$  is the identity  $I \in \mathbf{SO}(q)$ , and for  $0 \leq t \leq 2/3$  the action of  $\rho_{\ell,j,t}$  is multiplication by  $\overline{\text{exp}}(\vec{a}_{\ell,j})$ , hence  $\rho_{\ell,t}$  has image isomorphic to  $\mathfrak{G}_\ell$ .

A point  $\vec{v} \in \mathbb{R}^q$  is *generic* for  $\rho_\ell$  if  $\rho_\ell(\gamma) \cdot \vec{v} \neq \vec{v}$  for all  $\gamma \notin \Lambda_\ell$ . Let  $\mathcal{O}_\ell \subset \mathbb{R}^q$  denote the generic points for  $\rho_\ell$ . The fix-point set for an isometry  $\rho_\ell(\gamma)$  of  $\mathbb{R}^q$  is a proper subspace of  $\mathbb{R}^q$  if  $\gamma \notin \Lambda_\ell$ , and as  $\mathfrak{G}_\ell$  is a finite group, the set of non-generic points in  $\mathbb{R}^q$  is a finite union of proper subspaces. Thus,  $\mathcal{O}_\ell$  is an open and dense subset.

We are now ready to begin the inductive construction. For  $n = 0$ , we have:

$$\epsilon_0 = 1, \quad \rho_0 = I, \quad \Phi_0 = Id, \quad \Gamma_0 = \Lambda_0 = \mathbb{Z}^k, \quad N_0 = \mathbb{T}^k \times \mathbb{D}_{\epsilon_0}^q, \quad K_0 = \mathbb{D}_{\epsilon_0}^q$$

For consistency of notation with the subsequent inductive steps, set  $\widehat{N}_0 = N_0$  and let  $\Psi_0: N_0 \rightarrow \widehat{N}_0$  be the identity map. The foliation  $\mathcal{F}_0$  of  $N_0$  is the product foliation, and  $\widehat{\mathcal{F}}_0$  of  $\widehat{N}_0$  is the image of  $\mathcal{F}_0$  under  $\Psi_0$ . Then  $\widehat{L}_0 = \Psi_0(\mathbb{T}^k \times \vec{0}) \subset \widehat{N}_0$  is the leaf of  $\widehat{\mathcal{F}}_0$  through  $\vec{z}_0 = \vec{0}$ . Set  $\widehat{K}_0 = \Psi_0([\vec{0}] \times K_0)$ .

We next define the foliation  $\widehat{\mathcal{F}}_1$  of  $\widehat{N}_0$ . To begin, there is a continuous decomposition

$$(11) \quad N_0 = \bigcup_{0 \leq r \leq \epsilon_0} \mathbb{T}^k \times \mathbb{S}_r^{q-1}$$

For each  $0 \leq r \leq \epsilon_0$ , let  $\mathcal{F}_1$  restricted to  $\mathbb{T}^k \times \mathbb{S}_r^{q-1}$  be the foliation defined by the representation  $\rho_{1,t}$  where  $t = r/\epsilon_0$ . The foliation  $\mathcal{F}_1$  of  $N_0$  is smooth, as  $\rho_{1,t}$  depends smoothly on  $t$ .

Set  $\epsilon'_0 = 2/3 \cdot \epsilon_0$  and  $\epsilon''_0 = 3/4 \cdot \epsilon_0$ .

The family  $\{\rho_{1,t} \mid 0 \leq t \leq 1\}$  is an isotopy between  $\rho_1$  and the trivial representation  $\rho_0$ . The foliation  $\mathcal{F}_1$  restricted to  $\mathbb{T}^k \times (\mathbb{D}_{\epsilon_0}^q - \mathbb{B}_{\epsilon'_0}^q)$  is the product foliation;  $\mathcal{F}_1$  restricted to  $\mathbb{T}^k \times \mathbb{D}_{\epsilon'_0}^q$  equals  $\mathcal{F}_{\rho_1}$  whose holonomy is given by multiplication by the complex vectors  $\rho_1(\gamma) \in \mathbb{T}^m$ , for  $\gamma \in \mathbb{Z}^k$ . For  $\epsilon'_0 < r < \epsilon''_0$ , the foliation restricted to  $\mathbb{T}^k \times \mathbb{S}_r^{q-1}$  is the suspension of an isometric action.

Note that  $\mathcal{F}_1$  is a distal foliation. Given any two points  $z \neq z' \in \mathbb{S}_r^{q-1}$  the holonomy action of  $\mathcal{F}_1$  on these points is isometric, hence stays a bounded distance apart. On the other hand, if  $z \in \mathbb{S}_r^{q-1}$  and  $z' \in \mathbb{S}_{r'}^{q-1}$  for  $0 \leq r < r' \leq \epsilon_0$  then their orbits remain on distinct spherical shells, hence remain a bounded distance apart.

Let  $\widehat{\mathcal{F}}_1$  be foliation of  $\widehat{N}_0$  which is the image of  $\mathcal{F}_1$  under  $\Psi_0: N_0 \rightarrow \widehat{N}_0$ .

The inductive step for  $n = 1$  includes several further choices of data.

Let  $\vec{z}_1 \in \mathbb{S}_{\epsilon_0/2}^{q-1} \cap \mathcal{O}_1 \subset \mathbb{B}_{\epsilon'_0}^q \cap \mathcal{O}_1$  be a generic point for  $\rho_1$ . For  $\gamma \in \mathbb{Z}^k$  set  $\vec{z}_{1,\gamma} = \rho_1(\gamma)(\vec{z}_1)$ . The  $\rho_1$ -orbit  $\{\vec{z}_{1,\gamma} \mid \gamma \in \mathbb{Z}^k\}$  of  $\vec{z}_1$  is finite, so there exists  $\epsilon_1 > 0$  such that the closed disk centered at  $\vec{z}_1$  satisfies

$$(12) \quad \mathbb{D}_{\epsilon_1}^q(\vec{z}_1) \equiv \{\vec{z} \in \mathbb{R}^q \mid \|\vec{z} - \vec{z}_1\| \leq \epsilon_1\} \subset \mathbb{B}_{\epsilon'_0}^q \cap \mathcal{O}_1$$

and the translates under the action of  $\rho_1$  are disjoint. That is, if  $\gamma \in \mathbb{Z}^k$  satisfies  $\gamma \notin \Lambda_1$  then

$$\mathbb{D}_{\epsilon_1}^q(\vec{z}_1) \cap \rho_1(\gamma) \cdot \mathbb{D}_{\epsilon_1}^q(\vec{z}_1) = \mathbb{D}_{\epsilon_1}^q(\vec{z}_1) \cap \mathbb{D}_{\epsilon_1}^q(\vec{z}_{1,\gamma}) = \emptyset$$

Note that  $\|\vec{z}_1\| = \epsilon_0/2$  and  $\mathbb{D}_{\epsilon_1}^q(\vec{z}_1) \subset \mathbb{B}_{\epsilon'_0}^q \cap \mathcal{O}_1$  implies that  $\epsilon_1 < \epsilon_0/6$  and hence  $\mathbb{D}_{\epsilon_0/3}^q \cap \mathbb{D}_{\epsilon_1}^q(\vec{z}_1) = \emptyset$ . The finite union of the translates of  $\mathbb{D}_{\epsilon_1}^q(\vec{z}_1)$  under the action  $\rho_1$  is denoted by

$$(13) \quad K_1 = \bigcup_{\gamma \in \mathbb{Z}^k} \mathbb{D}_{\epsilon_1}^q(\vec{z}_{1,\gamma}) \subset K_0$$

Let  $N'_1$  denote the  $T_{\alpha_1}$ -saturation of the set  $[\vec{0}] \times \mathbb{D}_{\epsilon_1}^q(\vec{z}_1) \subset N_0$ . That is,

$$(14) \quad N'_1 = \bigcup_{\vec{\xi} \in \mathbb{R}^k} ([\vec{\xi}], \rho_1(\vec{\xi}) \cdot \mathbb{D}_{\epsilon_1}^q(\vec{z}_1)) \subset N_0$$

Finally, we put the foliation  $\mathcal{F}_1$  of  $N'_1$  into “standard form” in preparation for the next stage of the induction. Define a  $T_{\alpha_1}$ -equivariant map, for  $\xi \in \mathbb{R}^k$

$$(15) \quad \begin{aligned} \varphi_1: \mathbb{R}^k/\Lambda_1 \times \mathbb{D}_{\epsilon_1}^q &\longrightarrow N'_1 \\ \varphi_1([\vec{\xi}], \vec{z}) &= ([\vec{\xi}], \rho_1(\vec{\xi})(\vec{z}_1 + \vec{z})) \end{aligned}$$

This is well-defined precisely because  $\Lambda_1 = \ker \rho_1$ . Note that the product foliation on  $\mathbb{R}^k/\Lambda_1 \times \mathbb{D}_{\epsilon_1}^q$  is mapped by  $\varphi_1$  to the restriction of  $\mathcal{F}_1$  to  $N'_1$ .

Set  $N_1 = \mathbb{T}^k \times \mathbb{D}_{\epsilon_1}^q$ , and define the diffeomorphism

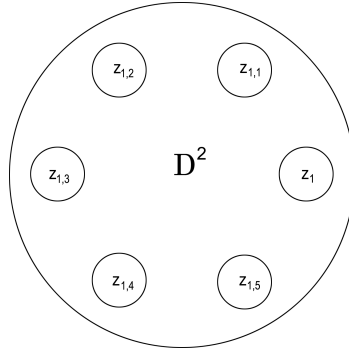
$$(16) \quad \begin{aligned} \phi_1: \mathbb{T}^k \times \mathbb{D}_{\epsilon_1}^q &\longrightarrow \mathbb{R}^k/\Lambda_1 \times \mathbb{D}_{\epsilon_1}^q \\ ([\vec{x}], [\vec{y}]) &\mapsto (\phi_1[\vec{x}], [\vec{y}]) \end{aligned}$$

Let  $\psi_1 = \varphi_1 \circ \phi_1: N_1 \rightarrow N'_1 \subset N_0$ . Set  $\Psi_1 = \Psi_0 \circ \psi_1: N_1 \rightarrow \hat{N}_0$ . The image of  $\Psi_1$  is denoted by  $\hat{N}_1$  which is a closed subset of  $\hat{N}_0$ . Set  $\hat{K}_1 = \hat{N}_1 \cap \hat{K}_0$ , which is the orbit of the disk  $[\vec{0}] \times \mathbb{D}_{\epsilon_1}^q \subset \hat{K}_0$  under the holonomy of  $\hat{\mathcal{F}}_1$ .

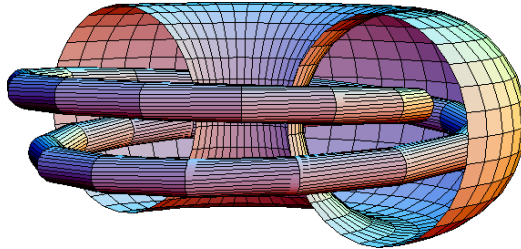
Note that  $\Psi_1$  maps the product foliation on  $N_1$  to the restriction of  $\hat{\mathcal{F}}_1$  to  $\hat{N}_1$ .

Let  $L_1 \subset N_1$  be the leaf which is the  $T_{\alpha_1}$ -orbit of  $[\vec{0}] \times \vec{z}_1$ , and  $\hat{L}_1 = \Psi_0(L_1) = \Psi_1(\mathbb{T}^k \times \vec{0})$ . Then  $\hat{L}_1$  is a leaf of  $\hat{\mathcal{F}}_1$  and  $\hat{N}_1$  is the closed  $\epsilon_1$ -disk bundle about  $\hat{L}_1$ .

Before proceeding onto the general inductive step, we give two illustrations of the constructions made above. First, for  $q = 2$  and  $\mathfrak{G}_1 = \mathbb{Z}/6\mathbb{Z}$  is cyclic of order 6, the following illustrates the set  $K_1$ , where the points  $z_{1,i}$  label the orbit of  $\vec{z}_1$  under  $\rho_1$ .



Next, again for  $q = 2$ , where  $\mathbb{T}^k = \mathbb{S}^1$  and  $\mathfrak{G}_1$  has order 2, the illustration below gives an idea of the set  $\hat{N}_1$ , which is an embedding of the solid 2-torus into itself.



Now assume that  $\hat{\mathcal{F}}_\ell$  of  $\hat{N}_0$  has been constructed, for  $\ell \geq 1$ . We construct the foliation  $\hat{\mathcal{F}}_{\ell+1}$  of  $\hat{N}_0$ .

Assume that choices of  $\{\vec{z}_1, \dots, \vec{z}_\ell\} \subset \mathbb{D}_{\epsilon_0}^q$  have been made, as well as the sequence  $1 > \epsilon_1 > \dots > \epsilon_\ell$  where  $\epsilon_{k+1} < \epsilon_k/6$  for  $1 \leq k < \ell$ . By convention, let  $\epsilon'_k = 2/3 \cdot \epsilon_k$  and  $\epsilon''_k = 3/4 \cdot \epsilon_k$  for  $1 \leq k \leq \ell$ .

By inductive hypotheses, we are given

$$(17) \quad \Psi_\ell = \Psi_{\ell-1} \circ \psi_\ell = \psi_1 \circ \dots \circ \psi_\ell: N_\ell = \mathbb{T}^k \times \mathbb{D}_{\epsilon_\ell}^q \rightarrow \hat{N}_\ell \subset \hat{N}_0$$



The foliation  $\widehat{\mathcal{F}}_\ell$  of  $\widehat{N}_0$  restricted to  $\widehat{N}_\ell$  is the image of the product foliation on  $N_\ell$ . The manifold with boundary  $\widehat{N}_\ell$  is the  $\ell^{th}$ -iteration of an embedding of a solid torus in the initial solid torus  $\widehat{N}_0$ .

We next define  $\mathcal{F}_{\ell+1}$  on  $N_\ell$  which is the product foliation on an open neighborhood of  $\partial N_\ell$ , then define  $\widehat{\mathcal{F}}_{\ell+1}$  to be  $\widehat{\mathcal{F}}_\ell$  on  $\widehat{N}_0 - \widehat{N}_\ell$ , and  $\widehat{\mathcal{F}}_{\ell+1}$  on  $\widehat{N}_\ell$  will be the image of  $\mathcal{F}_{\ell+1}$ . By the inductive hypothesis, the image under  $\Psi_\ell$  of the product foliation on  $N_\ell$  equals the restriction of  $\widehat{\mathcal{F}}_\ell$  on  $\widehat{N}_\ell$ , hence the image under  $\Psi_\ell$  of  $\mathcal{F}_{\ell+1}$  on  $\widehat{N}_\ell$  agrees with  $\widehat{\mathcal{F}}_\ell$  on an open neighborhood of  $\partial \widehat{N}_\ell$ , and so  $\widehat{\mathcal{F}}_{\ell+1}$  thus defined is a continuous (or possibly smooth) foliation of  $\widehat{N}_0$ .

To define  $\mathcal{F}_{\ell+1}$  on  $N_\ell$ , note there is a continuous decomposition

$$(18) \quad N_\ell = \bigcup_{0 \leq r \leq \epsilon_\ell} \mathbb{T}^k \times \mathbb{S}_r^{q-1}$$

For each  $0 \leq r \leq \epsilon_\ell$ , let  $\mathcal{F}_{\ell+1}$  restricted to  $\mathbb{T}^k \times \mathbb{S}_r^{q-1}$  be the foliation defined by the representation  $\rho_{\ell+1,t}$  where  $t = r/\epsilon_\ell$ . The foliation  $\mathcal{F}_{\ell+1}$  of  $N_\ell$  is smooth, as  $\rho_{\ell+1,t}$  depends smoothly on  $t$ .

The family  $\{\rho_{\ell+1,t} \mid 0 \leq t \leq 1\}$  is an isotopy between  $\rho_{\ell+1}$  and the trivial representation  $\rho_0$ . The foliation  $\mathcal{F}_{\ell+1}$  restricted to  $\mathbb{T}^k \times (\mathbb{D}_{\epsilon_\ell}^q - \mathbb{B}_{\epsilon_\ell''}^q)$  is the product foliation;  $\mathcal{F}_{\ell+1}$  restricted to  $\mathbb{T}^k \times \mathbb{D}_{\epsilon_\ell'}^q$  is the foliation  $\mathcal{F}_{\rho_{\ell+1}}$ , whose holonomy is given by multiplication by the complex vectors  $\rho_{\ell+1}(\gamma) \in \mathbb{T}^m$ , for  $\gamma \in \mathbb{Z}^k$ . For  $\epsilon_\ell' < r < \epsilon_\ell''$ , the foliation restricted to  $\mathbb{T}^k \times \mathbb{S}_r^{q-1}$  is the suspension of an isometric action. Thus, as before,  $\mathcal{F}_{\ell+1}$  is a distal foliation.

This completes the construction of  $\mathcal{F}_{\ell+1}$  on  $N_\ell$  and hence of  $\widehat{\mathcal{F}}_{\ell+1}$  on  $\widehat{N}_0$ . It remains to set up the remaining data for the induction.

Let  $\vec{z}_{\ell+1} \in \mathbb{S}_{\epsilon_\ell/2}^q \subset \mathbb{B}_{\epsilon_\ell'}^q \cap \mathcal{O}_{\ell+1}$  be a generic point for  $\rho_{\ell+1}$ .

For  $\gamma \in \mathbb{Z}^k$  set  $\vec{z}_{\ell+1,\gamma} = \rho_{\ell+1}(\gamma)(\vec{z}_{\ell+1})$ . The  $\rho_{\ell+1}$ -orbit  $\{\vec{z}_{\ell+1,\gamma} \mid \gamma \in \mathbb{Z}^k\}$  of  $\vec{z}_{\ell+1}$  is finite, so there exists  $\epsilon_{\ell+1} > 0$  such that the closed disk centered at  $\vec{z}_{\ell+1}$  satisfies

$$(19) \quad \mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1}) \subset \mathbb{B}_{\epsilon_\ell'}^q \cap \mathcal{O}_{\ell+1}$$

and the translates under the action of  $\rho_{\ell+1}$  are disjoint. Note that  $\mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1}) \subset \mathbb{B}_{\epsilon_\ell'}^q \cap \mathcal{O}_{\ell+1}$  implies that  $\epsilon_{\ell+1} < \epsilon_\ell/6$  and hence  $\mathbb{D}_{\epsilon_{\ell+1}}^q \cap \mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1}) = \emptyset$ . The finite union of the translates of  $\mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1})$  under the action  $\rho_{\ell+1}$  is denoted by

$$(20) \quad K_{\ell+1} = \bigcup_{\gamma \in \mathbb{Z}^k} \mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1,\gamma})$$

Let  $\#\mathfrak{G}_{\ell+1}$  denote the cardinality of the finite group  $\mathfrak{G}_{\ell+1}$ . Then  $K_{\ell+1}$  is the disjoint union of  $\#\mathfrak{G}_{\ell+1}$  closed disks, each of radius  $\epsilon_{\ell+1}$ . Observe that by construction,  $K_{\ell+1} \cap \mathbb{D}_{\epsilon_\ell/3}^q = \emptyset$ .

Let  $N'_{\ell+1}$  denote the  $T_{\alpha_{\ell+1}}$ -saturation of the set  $[\vec{0}] \times \mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1}) \subset N_\ell$ . That is,

$$(21) \quad N'_{\ell+1} = \bigcup_{\vec{\xi} \in \mathbb{R}^k} ([\vec{\xi}], \rho_{\ell+1}(\vec{\xi}) \cdot \mathbb{D}_{\epsilon_{\ell+1}}^q(\vec{z}_{\ell+1})) \subset N_\ell$$

Define a  $T_{\alpha_{\ell+1}}$ -equivariant map, for  $\xi \in \mathbb{R}^k$

$$(22) \quad \begin{aligned} \varphi_{\ell+1}: \mathbb{R}^k / \Lambda_{\ell+1} \times \mathbb{D}_{\epsilon_{\ell+1}}^q &\longrightarrow N'_{\ell+1} \\ \varphi_{\ell+1}([\vec{\xi}], \vec{z}) &= ([\vec{\xi}], \rho_{\ell+1}(\vec{\xi})(\vec{z}_{\ell+1} + \vec{z})) \end{aligned}$$

This is well-defined precisely because  $\Lambda_{\ell+1} = \ker \rho_{\ell+1}$ . The product foliation on  $\mathbb{R}^k / \Lambda_{\ell+1} \times \mathbb{D}_{\epsilon_{\ell+1}}^q$  is mapped by  $\varphi_{\ell+1}$  to the restriction of  $\mathcal{F}_{\ell+1}$  to  $N'_{\ell+1}$ .

Set  $N_{\ell+1} = \mathbb{T}^k \times \mathbb{D}_{\epsilon_{\ell+1}}^q$ , and define the diffeomorphism

$$(23) \quad \begin{aligned} \phi_{\ell+1}: \mathbb{T}^k \times \mathbb{D}_{\epsilon_{\ell+1}}^q &\longrightarrow \mathbb{R}^k / \Lambda_{\ell+1} \times \mathbb{D}_{\epsilon_{\ell+1}}^q \\ ([\vec{x}], [\vec{y}]) &\mapsto (\phi_{\ell+1}[\vec{x}], [\vec{y}]) \end{aligned}$$

Let  $\psi_{\ell+1} = \varphi_{\ell+1} \circ \phi_{\ell+1}: N_{\ell+1} \rightarrow N'_{\ell+1} \subset N_\ell$ . Set  $\Psi_{\ell+1} = \Psi_\ell \circ \psi_{\ell+1}: N_{\ell+1} \rightarrow \hat{N}_0$ . The image of  $\Psi_{\ell+1}$  is denoted by  $\hat{N}_{\ell+1}$  which is a closed subset of  $\hat{N}_0$ . Then  $\Psi_{\ell+1}$  maps to product foliation on  $N_{\ell+1}$  to the restriction of  $\hat{\mathcal{F}}_{\ell+1}$  to  $\hat{N}_{\ell+1}$ . Set  $\hat{K}_{\ell+1} = \hat{N}_{\ell+1} \cap \hat{K}_\ell = \hat{N}_{\ell+1} \cap \hat{K}_0$ .

Let  $L_{\ell+1} \subset N_{\ell+1}$  be the leaf of  $\mathcal{F}_{\ell+1}$  given by the  $T_{\alpha_{\ell+1}}$ -orbit of  $[\vec{0}] \times \vec{z}_{\ell+1}$ , and set  $\hat{L}_{\ell+1} = \Psi_\ell(L_{\ell+1}) = \Psi_{\ell+1}(\mathbb{T}^k \times \vec{0})$ . Then  $\hat{L}_{\ell+1}$  is a leaf of  $\hat{\mathcal{F}}_{\ell+1}$  and  $\hat{N}_{\ell+1}$  is a closed  $\epsilon_{\ell+1}$ -disk bundle about  $\hat{L}_{\ell+1}$ .

This completes the induction. Note that we obtain as a result:

- (1) a sequence of nested compact  $(k+q)$ -submanifolds with boundary,

$$\mathbb{T}^k \times \mathbb{D}_{\epsilon_0}^q \equiv \hat{N}_0 \supset \hat{N}_1 \supset \dots \supset \hat{N}_\ell \supset \dots$$

- (2) a sequence of nested compact  $q$ -submanifolds with boundary,  $\hat{K}_\ell = \hat{N}_\ell \cap \hat{K}_0$ , such that

$$\hat{K}_0 \supset \hat{K}_1 \supset \dots \supset \hat{K}_\ell \supset \dots$$

Moreover,  $\hat{K}_\ell$  is a union of  $\#\mathfrak{G}_\ell$  closed  $q$ -disks, each of radius  $\epsilon_\ell \leq \epsilon_0/6^{-\ell}$ .

- (3) a sequence of smooth foliations  $\hat{\mathcal{F}}_\ell$  of  $\hat{N}_0$  such that  $\hat{\mathcal{F}}_{\ell'} = \hat{\mathcal{F}}_\ell$  on  $\hat{N}_0 - \hat{N}_\ell$  for all  $\ell' > \ell$ .

The intersection  $\mathcal{S} = \bigcap_{\ell \geq 0} \hat{N}_\ell$  is homeomorphic to a generalized solenoid as defined in Definition 1.1, and the intersection  $\mathbb{K}_* = \bigcap_{\ell \geq 0} \hat{K}_\ell = \mathcal{S} \cap \hat{K}_0$  is a Cantor set.

Also note that as a result of the induction, we obtain a sequence of foliations  $\mathcal{F}_\ell$  of the manifolds  $N_\ell = \mathbb{T}^k \times \mathbb{D}_{\epsilon_\ell}^q$  where  $\mathcal{F}_\ell$  is the suspension of the family of representations  $\{\rho_{\ell+1,t} \mid 0 \leq t \leq 1\}$ . By definition,  $\rho_\ell$  is  $\delta_\ell$ -thin, and so the same holds for all representations  $\rho_{\ell+1,t}$  for  $0 \leq t \leq 1$ . By assumption,  $\delta_\ell \rightarrow 0$  as  $\ell \rightarrow \infty$ . Thus, the foliations  $\mathcal{F}_\ell$  limit to the product foliation of  $\mathbb{T}^k \times \mathbb{D}_{\epsilon_\ell}^q$ . It follows that there is a well-defined limit foliation  $\hat{\mathcal{F}}$  on  $\hat{N}_0$ , such that  $\hat{\mathcal{F}}$  restricted to  $\mathcal{S}$  agrees with the laminate structure of  $\mathcal{S}$  as a solenoid.

#### 4. $C^r$ -ESTIMATES

The foliations  $\hat{\mathcal{F}}_\ell$  constructed on  $\hat{N}_0 = \mathbb{T}^k \times \mathbb{D}^q$  in the last section are transverse to the factors  $[\vec{x}] \times \mathbb{D}^q$  for  $[\vec{x}] \in \mathbb{T}^k$ , and so can be alternately described in terms of their global holonomy maps, group actions  $h_\ell: \mathbb{Z}^k \rightarrow \text{Diff}^r(\mathbb{D}^q)$ , for  $r \geq 0$ , where we identify  $\mathbb{D}^q$  with  $[\vec{0}] \times \mathbb{D}^q = \hat{K}_0$ . In this section, we give explicit formulae for the maps  $h_{\ell,j} = h_\ell(\vec{e}_j)$ ,  $1 \leq j \leq k$ , which are the generators of the  $\mathbb{Z}^k$ -action. Using these explicit formulae, we can then give criteria on the maps to guarantee that for each  $j$ , the limit  $h_j = \lim_{\ell \rightarrow \infty} h_{\ell,j}$  is a  $C^r$ -diffeomorphism of  $\mathbb{D}^q$ .

The description of the holonomy maps  $h_{\ell+1,j}$  in terms of  $h_{\ell,j}$  for  $\ell > 1$  requires that we take into account the role of holonomy of  $\hat{\mathcal{F}}_\ell$  when defining  $\hat{\mathcal{F}}_{\ell+1}$ . In the inductive construction of  $\hat{\mathcal{F}}_{\ell+1}$ , the only modification is for  $\hat{\mathcal{F}}_\ell$  restricted to  $\hat{N}_\ell$ , and thus their holonomy maps agree on  $\hat{K}_0 - \hat{K}_\ell$ . To show that the limit  $h_j$  of  $\{h_{\ell,j}\}_{\ell=1}^\infty$  is  $C^r$ , it thus suffices to give  $C^r$ -estimates for  $h_{\ell+1,j} - h_{\ell,j}$  on  $\hat{K}_\ell$  which are sufficient to imply the sequence is Cauchy in the uniform  $C^r$ -topology.

The first step in obtaining these estimates to give explicit descriptions of the closed subsets  $\hat{K}_\ell \subset \hat{K}_0$ , and corresponding explicit formulae for the holonomy of  $\hat{\mathcal{F}}_{\ell+1}$  induced on  $\hat{K}_\ell$ . The estimates of derivatives of differences  $h_{\ell+1,j} - h_{\ell,j}$  on  $\hat{K}_\ell$  then follow from calculus.

The foliations  $\hat{\mathcal{F}}_{\ell+1}$  are defined on the manifolds  $N_\ell$  and then mapped into  $\hat{N}_0 = \mathbb{T}^k \times \mathbb{D}^q$  via the map  $\Psi_\ell$ , which “twists” the product foliation on  $N_\ell$  so that it agrees with the restriction of  $\hat{\mathcal{F}}_\ell$ . This foliated surgery obscures the role of  $\hat{\mathcal{F}}_\ell$  in the construction of  $\hat{\mathcal{F}}_{\ell+1}$ . This role must now be made explicit, which involves extending all of the holonomy generators of  $\hat{\mathcal{F}}_\ell$  to maps of the fundamental group of  $\hat{N}_0$  and not just for  $N_\ell$ . That is, we must extend these maps from being defined on  $\Gamma_\ell \subset \mathbb{Z}^k$  to all of  $\mathbb{Z}^k$ . We do this as follows.

For each  $\ell \geq 1$ , recall the choice of representation  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$  defines vectors  $\alpha_\ell(\vec{e}_j) = \vec{a}_{\ell,j} \in \mathbb{Q}^m$ .

The map  $\Phi_\ell: \mathbb{Z}^k \rightarrow \Gamma_\ell \subset \mathbb{Z}^k$  of Definition 3.1 extends to an isomorphism  $\Phi_\ell: \mathbb{Q}^k \rightarrow \mathbb{Q}^k$ , and hence has inverse  $\Phi_\ell^{-1}: \mathbb{Q}^k \rightarrow \mathbb{Q}^k$ , which restricts to a map  $\Phi_\ell^{-1}: \mathbb{Z}^k \rightarrow \mathbb{Z}^k$ . The isomorphism  $\Phi_\ell^{-1}$  is represented by a matrix with rational entries, with common denominator  $\det(\Phi_\ell) = \#(\mathfrak{G}_1 \times \cdots \times \mathfrak{G}_\ell)$ .

For each  $\ell \geq 1$ ,  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$  admits a unique extension  $\alpha_\ell: \mathbb{Q}^k \rightarrow \mathbb{Q}^m$ , and we set  $\alpha'_\ell = \alpha_\ell \circ \Phi_{\ell-1}^{-1}$ . This represents the additional “rotation” on  $\widehat{K}_\ell$  introduced in modifying  $\widehat{\mathcal{F}}_{\ell-1}$  to obtain  $\widehat{\mathcal{F}}_\ell$ . The cumulative rotation of the holonomy of  $\widehat{\mathcal{F}}_\ell$ , for  $\ell \geq 1$ , is thus defined by maps  $\beta_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$

$$(24) \quad \beta_1 = \alpha'_1, \beta_2 = \alpha'_1 + \alpha'_2, \dots, \beta_\ell = \alpha'_1 + \alpha'_2 + \cdots + \alpha'_\ell$$

Set  $\beta_{\ell,j} = \beta_\ell(\vec{e}_j) \in \mathbb{Q}^m$  for  $1 \leq j \leq k$ . Let  $\Upsilon_\ell(\gamma) = \overline{\text{exp}}(\beta_\ell(\gamma))$  for  $\gamma \in \mathbb{Z}^k$ , and  $\Upsilon_{\ell,j} = \overline{\text{exp}}(\beta_{\ell,j})$ .

The explicit description of the subset  $\widehat{K}_\ell \subset \mathbb{D}^q$ , which is a disjoint union of disks of radius  $\epsilon_\ell$ , requires a formula for the centers  $\{\widehat{z}_{\ell,\gamma} \mid \gamma \in \mathbb{Z}^k\}$  of these disks. To begin,  $\widehat{z}_0 = \vec{0}$ , and  $\rho_1$  acts via rotation about  $\widehat{z}_0 = \widehat{z}_{0,\gamma}$ . Given the choice  $\vec{z}_1 \in \mathbb{D}^q$  with  $\|\vec{z}_1\| = \epsilon_0/2$ , we have

$$\widehat{z}_{1,\gamma} = z_{1,\gamma} = \rho_1(\gamma)(\vec{z}_1) = \overline{\text{exp}}(\alpha'_1(\gamma)) \cdot \vec{z}_1 \in \mathbb{B}_{\epsilon_0}^q, \quad \gamma \in \Gamma_0 = \mathbb{Z}^k$$

In general, the action of  $\rho_\ell(\gamma)$  acts via rotations on the disk  $\mathbb{D}_{\epsilon_{\ell-1}}^q$  and having chosen the generic point  $\vec{z}_\ell \in \mathbb{B}_{\epsilon_{\ell-1}}^q$  with  $\|\vec{z}_\ell\| = \epsilon_{\ell-1}/2$ , we have

$$(25) \quad \vec{z}_{\ell,\gamma} = \rho_\ell(\Phi_{\ell-1}^{-1}(\gamma))(\vec{z}_\ell) = \overline{\text{exp}}(\alpha'_\ell(\gamma)) \cdot \vec{z}_\ell, \quad \gamma \in \Gamma_{\ell-1} \subset \mathbb{Z}^k$$

We use (25) to define  $\vec{z}_{\ell,\gamma}$  for all  $\gamma \in \mathbb{Z}^k$ . Note that for  $\delta \in \Gamma_\ell$  we have that  $\vec{z}_{\ell,\gamma+\delta} = \vec{z}_{\ell,\gamma}$  and in particular,  $\vec{z}_{\ell,\delta} = \vec{z}_{\ell,0} = \vec{z}_\ell$ .

The point  $\widehat{z}_{\ell,\gamma}$  is obtained from the point  $\vec{z}_{\ell,\gamma}$  by composition with the map  $\Psi_{\ell-1}$  of (17), thus for all  $\gamma \in \mathbb{Z}^k$  we have  $\widehat{z}_{\ell,\gamma} = \widehat{z}_{\ell-1,\gamma} + \overline{\text{exp}}(\beta_\ell(\gamma)) \cdot \vec{z}_\ell$ . Hence, by induction we have

$$(26) \quad \widehat{z}_{\ell,\gamma} = \overline{\text{exp}}(\beta_1(\gamma)) \cdot \vec{z}_1 + \cdots + \overline{\text{exp}}(\beta_\ell(\gamma)) \cdot \vec{z}_\ell$$

The orbit  $\{\widehat{z}_{\ell,\gamma} \mid \gamma \in \mathbb{Z}^k\}$  corresponds to the centers of the disks in  $\widehat{K}_\ell$  so that

$$(27) \quad \widehat{K}_\ell = \bigcup_{\gamma \in \mathbb{Z}^k} \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma}) \subset \widehat{K}_0$$

For each  $\gamma \in \mathbb{Z}^k$ , define

$$(28) \quad \lambda_{\ell,\gamma}: \mathbb{R}^q \rightarrow \mathbb{R}^q, \quad \lambda_{\ell,\gamma}(\vec{z}) = \widehat{z}_{\ell,\gamma} + \epsilon_\ell \cdot \vec{z}$$

Note that if  $\delta \in \Gamma_\ell$  then  $\lambda_{\ell,\delta+\gamma} = \lambda_{\ell,\gamma}$ , so that the collection  $\{\lambda_{\ell,\gamma}: \mathbb{D}^q \rightarrow \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma}) \mid \gamma \in \mathbb{Z}^k/\Gamma_\ell\}$  parametrizes the disks in  $\widehat{K}_\ell$ .

The perturbations used throughout the construction are based on a “standard model”: given  $\vec{a} \in \mathbb{R}^m$ , define  $g_{\vec{a}}: \mathbb{R}^q \rightarrow \mathbb{R}^q$  by

$$(29) \quad g_{\vec{a}}(\vec{z}) = \begin{cases} \overline{\text{exp}}(\vec{a}) \cdot \vec{z} & \text{if } 0 \leq t \leq 2/3 \\ \overline{\text{exp}}(\mu(t) \cdot \vec{a}) \cdot \vec{z} & \text{if } 2/3 < t < 3/4, \quad t = \|\vec{z}\| \\ \vec{z} & \text{if } 3/4 \leq t \end{cases}$$

Note that  $g_{\vec{a}}$  is the identity outside of  $\mathbb{D}_{3/4}^q$ , and is the constant “rotation” by  $\overline{\text{exp}}(\vec{a})$  on  $\mathbb{D}_{2/3}^q$ .

The definition of  $\rho_{1,t}$  in (10) is the same as specifying that, for  $\vec{z} \in \widehat{K}_0 = \mathbb{D}^q$ ,

$$(30) \quad h_{1,j}(\vec{z}) = \lambda_0 \circ g_{\vec{a}_{1,j}} \circ \lambda_0^{-1}(\vec{z}) = g_{\vec{a}_{1,j}}(\vec{z})$$

Set  $U_0 = \mathbb{B}_{\epsilon_0/6}^q$ , then for  $\vec{z} \in U_1$ ,  $h_{1,j}(\vec{z}) = \overline{\text{exp}}(\vec{a}_{1,j}) \cdot \vec{z}$ .

The composition  $\lambda_{\ell,\gamma} \circ g_{\alpha'_{\ell+1}(\gamma)} \circ \lambda_{\ell,\gamma}^{-1}: \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma}) \rightarrow \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma})$  is rotation by  $\overline{\text{exp}}(\alpha'_{\ell+1}(\gamma))$  around  $\widehat{z}_{\ell,\gamma}$ .

Define  $\widehat{h}_{\ell+1}(\gamma): \widehat{K}_\ell \rightarrow \widehat{K}_\ell$  as follows: given  $\vec{z} \in \widehat{K}_\ell$ , then  $\vec{z} \in \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma})$  for some  $\gamma \in \mathbb{Z}^k/\Gamma_\ell$ , and set

$$(31) \quad \widehat{h}_{\ell+1}(\gamma)(\vec{z}) = \lambda_{\ell,\gamma} \circ g_{\alpha'_{\ell+1}(\gamma)} \circ \lambda_{\ell,\gamma}^{-1}(\vec{z})$$

For  $\gamma \in \Gamma_\ell$  and  $\vec{z} \in \mathbb{D}_{\epsilon_\ell}^q(\widehat{z}_{\ell,\gamma})$ , note that

$$(32) \quad \widehat{h}_{\ell+1}(\gamma)(\vec{z}) = \rho_{\ell+1}(\Phi_\ell^{-1}(\gamma))(\vec{z} - \widehat{z}_{\ell,\gamma}) + \widehat{z}_{\ell,\gamma}$$

so that  $\widehat{h}_{\ell+1}(\gamma)$  is the holonomy of  $\widehat{\mathcal{F}}_{\ell+1}$  introduced by the foliated surgery of section 3. That is, for  $\gamma \in \mathbb{Z}^k$  and  $\vec{z} \in \widehat{K}_\ell$ ,

$$(33) \quad h_{\ell+1}(\gamma)(\vec{z}) = \widehat{h}_{\ell+1}(\gamma) \circ h_\ell(\gamma)(\vec{z})$$

Note that for  $\vec{a} \in \mathbb{R}^m$ , the restriction of  $\lambda_{\ell,\gamma} \circ g_{\vec{a}} \circ \lambda_{\ell,\gamma}^{-1}$  to  $\mathbb{D}_{\epsilon'_\ell}^q(\widehat{z}_{\ell,\gamma})$  is multiplication by  $\overline{\text{exp}}(\vec{a})$ . Moreover,  $\mathbb{B}_{\epsilon_\ell/6}^q(\widehat{z}_{\ell,\gamma}) \cap \mathbb{B}_{\epsilon_{\ell+1}}^q(\widehat{z}_{\ell+1,\gamma}) = \emptyset$  for all  $\gamma$ . Set

$$(34) \quad U_\ell = \bigcup_{\gamma \in \mathbb{Z}^k} \mathbb{B}_{\epsilon_\ell/6}^q(\widehat{z}_{\ell,\gamma}) \subset \widehat{K}_\ell \subset \widehat{K}_0$$

Then the restriction of  $h_{\ell+1}(\gamma)$  to  $U_\ell$  acts via linear isometries, and is the identity for  $\gamma \in \Gamma_{\ell+1}$ . It follows that  $h_{\ell'}$  defines an affine action of finite order on  $U_\ell$  for all  $\ell, \ell' \geq 0$ ,

It remains to estimate  $h_{\ell+1,j} - h_{\ell,j} = h_{\ell+1}(\vec{e}_j) - h_\ell(\vec{e}_j)$  in the  $C^r$ -topology on  $\widehat{K}_\ell \subset \mathbb{D}^q$ . The first reduction is to observe, as noted above, that if  $\vec{z} \notin \widehat{K}_\ell$ , then  $h_{\ell+1,j}(\vec{z}) - h_{\ell,j}(\vec{z}) = 0$ . Secondly, for  $\vec{z} \in \widehat{K}_\ell$ , then

$$h_{\ell+1,j}(\vec{z}) - h_{\ell,j}(\vec{z}) = \{\widehat{h}_{\ell+1}(\vec{e}_j) - Id\} \circ h_{\ell,j}(\vec{z})$$

where  $h_{\ell,j}$  acts via locally constant linear isometries on  $\widehat{K}_\ell$ , hence the  $C^r$ -norm of  $h_{\ell+1,j} - h_{\ell,j}$  and that of  $\{\widehat{h}_{\ell+1}(\vec{e}_j) - Id\}$  on  $\widehat{K}_\ell$  are equal.

By (27) it suffices to estimate the  $C^r$ -norm of  $\{\widehat{h}_{\ell+1}(\vec{e}_j) - Id\}$  on each disk  $\mathbb{D}_{\epsilon_\ell}^q(\vec{z}_{\ell,\gamma})$ , where  $\gamma \in \mathbb{Z}^k$ . Note that from (31) we see that  $\{\widehat{h}_{\ell+1}(\vec{e}_j) - Id\}$  is a conjugate of the basic map  $g_{\alpha'_{\ell+1}(\vec{e}_j)}$  by the affine map  $\lambda_{\ell,\gamma}$  where the latter has derivative  $\epsilon_\ell$ . By the Chain Rule, it suffices to estimate  $\{g_{\alpha'_{\ell+1}(\vec{e}_j)} - Id\}$  on  $\mathbb{D}^q$  in the  $C^r$ -norm, and then scale the result by  $\epsilon_\ell^{1-r}$ . Finally, note that  $g_{\alpha'_{\ell+1}(\vec{e}_j)}$  restricted to  $\mathbb{D}_{2/3}^q$  is multiplication by  $\overline{\text{exp}}(\alpha'_{\ell+1}(\vec{e}_j))$  so all of its higher derivatives vanish near the origin. Thus, for  $\vec{a} \in \mathbb{R}^m$ , it remains to estimate the  $C^r$ -norm of the smooth map  $\{g_{\vec{a}} - Id\}$  on  $\mathbb{D}^q$ . We obtain an estimate which depends linearly on  $\|\vec{a}\|$ . Hence, with appropriate choices of  $\vec{a} = \alpha_{\ell,j}$  this will imply that the sequence of maps  $\{h_\ell\}$  is Cauchy in the  $C^r$ , or even,  $C^\infty$ -topology.

Fix  $0 \neq \vec{z} \in \mathbb{D}^q$ , with  $r = \|\vec{z}\|$ , and let  $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_q\}$  be an orthonormal basis of  $T_{\vec{z}}\mathbb{D}^q \cong \mathbb{R}^q$  such that  $\vec{u}_1 = \frac{1}{r}\partial/\partial r$  points in the radial direction.

For  $\vec{v} \in \mathbb{R}^q$ , let  $\nabla_{\vec{v}}$  denote the partial derivative operator on functions  $f: \mathbb{D}^q \rightarrow \mathbb{R}^q$ .

First note that for  $1 < i \leq q$ ,  $\nabla_{\vec{u}_i}(g_{\vec{a}}) = \rho_{\vec{a},t} \cdot \vec{u}_i$  which depends only on the variable  $t$ .

On the other hand, for  $\vec{v} = \vec{u}_1$  by the chain rule we have, for  $t = \|\vec{z}\|$ ,

$$(35) \quad \nabla_{\vec{u}_1}(g_{\vec{a}}) = \rho_{\vec{a},t} \cdot \vec{u}_1 + 2\pi\mu'(t) \cdot \vec{a} \cdot \rho_{\vec{a},t} \cdot \vec{z}$$

Thus, for all  $1 \leq i \leq q$ , we have

$$(36) \quad \|\nabla_{\vec{u}_i}(g_{\vec{a}} - Id)\| \leq \|\overline{\text{exp}}(\vec{a}) - \overline{\text{exp}}(\vec{0})\| + 2\pi\|\vec{a}\| \cdot \kappa_1$$

where  $\kappa_1 = \max\{|\mu'(t)| \mid 0 \leq t \leq 1\}$ . Note that  $\|\overline{\text{exp}}(\vec{a}) - \overline{\text{exp}}(\vec{0})\| \leq \|\vec{a}\|$ . Thus, by (31) and the previous reductions, we obtain an estimate on the uniform  $C^1$ -norm  $\|\cdot\|_{(1)}$ :

$$(37) \quad \|\widehat{h}_{\ell+1}(\vec{e}_j) - Id\|_{(1)} \leq \epsilon_\ell \cdot (1 + 2\pi\kappa_1) \cdot \epsilon_\ell^{-1} \cdot \|\alpha'_{\ell+1}(\vec{e}_j)\| = (1 + 2\pi\kappa_1) \cdot \|\alpha'_{\ell+1}(\vec{e}_j)\|$$

The estimate on the uniform  $C^p$ -norm  $\|g_{\vec{a}} - Id\|_{(p)}$  for  $p > 1$  is more complicated, and involves the higher order derivatives of  $\mu(t)$ . Some reductions of the calculation are possible.

For second order partials, where  $p = 2$ , if  $1 < i_1 \leq i_2 \leq q$ , then we saw  $\nabla_{\vec{u}_{i_1}}(g_{\vec{a}}) = \rho_{\vec{a},t} \cdot \vec{u}_{i_1}$ , hence

$$\begin{aligned} \nabla_{\vec{u}_{i_2}} \circ \nabla_{\vec{u}_{i_1}}(g_{\vec{a}} - Id) &= 0 \\ \nabla_{\vec{u}_{i_1}} \circ \nabla_{\vec{u}_1}(g_{\vec{a}} - Id) &= (2\pi\mu'(t) \cdot \vec{a} \cdot \overline{\exp}(\vec{a}) \cdot \vec{u}_{i_1}) \\ &= \vec{a} \cdot (2\pi\mu'(t) \cdot \overline{\exp}(\vec{a}) \cdot \vec{u}_{i_1}) \end{aligned}$$

while for the second derivative in the radial direction we have

$$\begin{aligned} \nabla_{\vec{u}_1} \circ \nabla_{\vec{u}_1}(g_{\vec{a}} - Id) &= 2\pi\mu'(t) \cdot \vec{a} \cdot \rho_{\vec{a},t} \cdot \vec{u}_1 + 4\pi\mu'(t)^2 \cdot \vec{a} \cdot \vec{a} \cdot \rho_{\vec{a},t} \cdot \vec{u}_1 + 2\pi\mu''(t) \cdot \vec{a} \cdot \rho_{\vec{a},t} \cdot \vec{z} \\ &= \vec{a} \cdot (2\pi\mu'(t) \cdot \rho_{\vec{a},t} \cdot \vec{u}_1 + 4\pi\mu'(t)^2 \cdot \vec{a} \cdot \rho_{\vec{a},t} \cdot \vec{u}_1 + 2\pi\mu''(t) \cdot \rho_{\vec{a},t} \cdot \vec{z}) \end{aligned}$$

Let  $1 \leq i_1 \leq \dots \leq i_p \leq q$ , and consider the expression:

$$(38) \quad \nabla_{\vec{u}_{i_p}} \circ \dots \circ \nabla_{\vec{u}_{i_2}} \circ \nabla_{\vec{u}_{i_1}}(g_{\vec{a}} - Id)$$

If two or more indices in (38) are greater than 1, then the expression vanishes as partial derivatives commute. Otherwise, the expression (38) always has the vector factor  $\vec{a}$  multiplying an expression whose scalar terms involve the derivatives of  $\mu(t)$  up to order  $\nu = p - 1$ . Hence, there exists  $C_p > 0$  independent of  $\vec{a}$ , such that

$$(39) \quad \|\nabla_{\vec{u}_{i_p}} \circ \dots \circ \nabla_{\vec{u}_{i_2}} \circ \nabla_{\vec{u}_{i_1}}(g_{\vec{a}} - Id)\| \leq \|\vec{a}\| \cdot C_p$$

Thus, by (31) and the previous reductions, we obtain an estimate on the uniform  $C^p$ -norm:

$$(40) \quad \|\widehat{h}_{\ell+1}(\vec{e}_j) - Id\|_{(p)} \leq C_p \cdot \epsilon_\ell^{1-p} \cdot \|\alpha'_{\ell+1}(\vec{e}_j)\|$$

Note that the constants  $C_p$  may tend to infinity rapidly. Moreover, the diameters satisfy  $\epsilon_\ell \leq 1/6^\ell$ , hence for  $p > 1$  the term  $\epsilon_\ell^{1-p}$  grows exponentially fast.

Given  $\delta > 0$  and an integer  $1 \leq r \leq \infty$ , we require conditions such that for each  $1 \leq j \leq k$ , the sequence of maps  $\{h_{\ell,j}\}_{\ell=1}^\infty$  is Cauchy in the  $C^p$ -topology, for all  $1 \leq p \leq r$ . For example, require that for all  $\ell \geq 1$  and all  $1 \leq p \leq r$ ,

$$(41) \quad \|\nabla_{\vec{u}_{i_p}} \circ \dots \circ \nabla_{\vec{u}_{i_2}} \circ \nabla_{\vec{u}_{i_1}}(g_{\alpha'_{\ell+1}(\vec{e}_j)} - Id)\| \leq \delta 2^{-\ell}$$

then by the above remarks we have that the sequence  $\{h_{\ell,j}\}_{\ell=1}^\infty$  converges to a  $C^r$  map  $h_j$  such that

$$\|\nabla_{\vec{u}_{i_p}} \circ \dots \circ \nabla_{\vec{u}_{i_2}} \circ \nabla_{\vec{u}_{i_1}}(h_j - Id)\| \leq \delta$$

If we require that (41) holds for all  $\ell \geq 1$  and all  $1 \leq p \leq \ell$ , then the limit  $h_j$  will be  $C^\infty$ .

The estimates (37) and (40) show this holds if for each  $1 \leq p \leq r$  we require that

$$(42) \quad C_p \cdot \epsilon_\ell^{1-p} \cdot \|\alpha'_{\ell+1}(\vec{e}_j)\| \leq \delta 2^{-\ell}, \text{ or } \|\alpha'_{\ell+1}(\vec{e}_j)\| \leq \delta \epsilon_\ell^{r-1} / 2^\ell C_p$$

It remains to note that  $\alpha'_{\ell+1} = \alpha_{\ell+1} \circ \Phi_\ell^{-1}$  so

$$(43) \quad \|\alpha'_{\ell+1}(\vec{e}_j)\| \leq \|\alpha_{\ell+1}\| \cdot \|\Phi_\ell^{-1}\| \leq \delta_\ell \cdot \|\Phi_\ell^{-1}\|$$

As remarked above, the map  $\Phi_\ell$  depends upon the choices of the vector  $\vec{a}_{k,j} \in \mathbb{Q}^m$  for  $1 \leq k \leq \ell$ , so (42) and (43) gives the inductive estimate

$$(44) \quad \delta_{\ell+1} \leq \delta \epsilon_\ell^{r-1} / 2^\ell C_p \|\Phi_\ell^{-1}\|$$

If the representations  $\alpha_\ell: \mathbb{Z}^k \rightarrow \mathbb{Q}^m$  are chosen so that (44) holds, then the limit  $\widehat{\mathcal{F}}$  is  $C^r$ .

## 5. AN EXAMPLE

In this section, we consider the above is a recipe for constructing a  $C^r$ -diffeomorphism  $h: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  whose suspension flow contains a solenoidal minimal set. This is the most familiar and intuitive case, for which there is an extensive literature (see, for example [4, 9, 10, 15, 20, 30]). Of course, the main point of this paper is to give an explicit construction which yields higher dimensional solenoids, but examining this simplest case illustrates the more obscure steps of the induction in sections 3 and 4.

Let  $k = 1$  and  $q = 2$ , and let  $r \geq 1$ .

The first step is to choose representations  $\alpha_\ell: \mathbb{Z} \rightarrow \mathbb{Q}$ . We set  $\alpha_\ell(u) = u/n_\ell$  for integers  $n_\ell > 1$ . Then  $\Lambda_\ell = n_\ell \cdot \mathbb{Z}$ , and  $\mathfrak{G}_\ell = (\frac{1}{n_\ell} \cdot \mathbb{Z})/\mathbb{Z}$ . Then  $e_1^\ell = +n_\ell$  is the unique choice for the oriented generator of  $\Lambda_\ell$ . The map  $\phi_\ell(u): \mathbb{R} \rightarrow \mathbb{R}$  is given by  $\phi_\ell(r) = n_\ell \cdot r$ , inducing the diffeomorphism  $\bar{\phi}_\ell: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/n_\ell\mathbb{Z}$ . The map  $\Phi_\ell: \mathbb{Z} \rightarrow \mathbb{Z}$  is then given by  $\Phi_\ell(u) = n_1 \cdots n_\ell \cdot u$ .

The inverse  $\Phi_\ell: \mathbb{Q} \rightarrow \mathbb{Q}$  is given by  $\Phi_\ell^{-1}(\frac{u}{v}) = \frac{u}{n_1 \cdots n_\ell \cdot v}$ . Then

$$\alpha'_\ell(u) = \alpha_\ell \circ \Phi_{\ell-1}^{-1}(u) = \frac{u}{n_\ell} \cdot \frac{1}{n_1 \cdots n_{\ell-1}} = \frac{u}{n_1 \cdots n_\ell}$$

It happens that  $\alpha'_\ell(u) = \Phi_\ell^{-1}(u)$  as the choices of generators for each  $\Lambda_\ell$  were unique. Then

$$(45) \quad \beta_\ell(u) = u \cdot \left\{ \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_\ell} \right\} = u \cdot \beta_{\ell,1}$$

The choices of the radii  $\epsilon_\ell$  are dictated by the conditions (19). If we choose the generic point  $\vec{v}_\ell \in \mathbb{D}_{\epsilon_\ell}^2$  with  $\|\vec{v}_\ell\| = \epsilon_\ell/2$ , then the distance between the translates of  $\vec{v}_\ell$  by the rotation group of order  $n_\ell$  will be  $\epsilon_\ell \sin(\pi/n_\ell)$ . Since  $\sin(\pi/n_\ell) \geq 1/n_\ell$  for all  $n_\ell > 1$  we can choose  $\epsilon_{\ell+1} = \epsilon_\ell/4n_\ell$  and satisfy the disjoint translates condition, as well as have  $\mathbb{D}_{\epsilon_{\ell+1}}^2 \subset \mathbb{D}_{\epsilon_\ell}^2$ . That is, we can let

$$\epsilon_\ell = \frac{1}{4^\ell n_1 \cdots n_\ell}$$

Let  $h_\ell = \widehat{h}_\ell(\vec{e}_1)$ . Then the estimate on the first derivatives (37) becomes

$$(46) \quad \|\widehat{h}_{\ell+1} - Id\|_{(1)} \leq \frac{(1 + 2\pi\kappa_1)}{n_1 \cdots n_{\ell+1}}$$

where  $\kappa_1 = \max\{|\mu'(t)| \mid 0 \leq t \leq 1\}$ .

Given  $\delta > 0$ , the  $C^1$ -norm of  $h$  satisfies  $\|h\|_{(1)} < \delta$  if  $n_1 > 2(1 + 2\pi\kappa_1)/\delta$ , and all subsequent  $n_\ell > 2$ .

The estimate for higher derivatives is more involved, as it depends on the  $C^r$ -norm of  $\mu$  and the radii  $\epsilon_\ell$ . From the estimates (37), (40), (42) and (43), it suffices for each  $1 \leq p \leq r$  that we choose  $n_{\ell+1}$  to satisfy

$$(47) \quad \delta_{\ell+1} \leq \delta \epsilon_\ell^{p-1} / 2^\ell C_p \|\Phi_\ell^{-1}\|, \text{ or } n_{\ell+1} \geq \frac{C_p \cdot (4^\ell n_1 \cdot n_\ell)^p}{\delta \cdot 2^\ell}$$

Clearly, the orders  $\{n_\ell\}$  of the holonomy groups of the inserted disks tend to infinity quite rapidly.

It is worth noting that for any given  $\delta > 0$ , there are an uncountable number of sequences  $\{n_\ell\}_{\ell=1}^\infty$  which satisfy the inductive criteria (47).

## 6. REMARKS AND QUESTIONS

Our construction leads to a solenoid that occurs as a minimal set of a distal  $C^r$ -foliation. In fact, we can apply the same construction to generate a distal  $C^r$ -foliation that contains a countably infinite family of minimal solenoids, no two of which are homeomorphic.

To see this, first observe that for a solenoid  $\mathcal{S}$  as presented in Definition 1.1, the  $n$ -th Čech cohomology group  $\check{H}^n(\mathcal{S}, \mathbb{Z})$  is isomorphic to the direct limit  $\varinjlim \{d_\ell: \mathbb{Z} \rightarrow \mathbb{Z}\}$ ,  $\ell \geq 1$ , where  $d_\ell$  is the degree of the

covering map  $p_\ell$ . This is a torsion-free group of rank one. Without loss of generality, one may assume that any such a group is presented as  $\varinjlim \{p_\ell: \mathbb{Z} \rightarrow \mathbb{Z}\}$ ,  $\ell \geq 1$ , where each  $p_\ell$  is a prime. According to Baer's classification of such groups [2, 17], two such groups determined by the sequences of primes  $P = (p_1, p_2, \dots)$  and  $Q = (q_1, q_2, \dots)$  are isomorphic if and only if it is possible to remove finitely many of the terms from each of the sequences  $P$  and  $Q$  to obtain new sequences  $P'$  and  $Q'$  in such a way that each prime number occurs with the same cardinality in  $P'$  and  $Q'$ , see e.g. [21]. Thus, by choosing these degrees appropriately, it is clear that there is an uncountable number of topologically distinct solenoids based on a given  $L_0$  as in Theorem 1.3. Also, using the portion of the plug that still has a product structure at the end of the original construction, we can insert in our original plug a countably infinite number of solenoids, no two of which are homeomorphic, and in such a way that the resulting foliation is distal and  $C^r$ .

In the following,  $L_0$  denotes a closed oriented manifold of dimension  $n \geq 1$  with  $H^1(L_0, \mathbb{R}) \neq 0$ , and  $M = L_0 \times \mathbb{D}^q$  is the product disk bundle over  $L_0$  for  $q \geq 2$ .

**QUESTION 6.1.** *Let  $r \geq 0$ , and  $\mathcal{F}, \mathcal{F}'$  denote  $C^r$ -foliations of  $M$  obtained as above, with solenoidal minimal sets  $\mathcal{S}, \mathcal{S}'$ . If  $\mathcal{S}$  and  $\mathcal{S}'$  are topologically conjugate solenoids, when are the foliations  $\mathcal{F}$  and  $\mathcal{F}'$   $C^r$ -conjugate? That is, when is there a  $C^r$ -diffeomorphism  $f: M \rightarrow M$  which maps leaves of  $\mathcal{F}$  to leaves of  $\mathcal{F}'$ , and hence maps  $\mathcal{S}$  to  $\mathcal{S}'$ ?*

Markus and Meyer have shown [20] that the generic Hamiltonian flow contains solenoids of all possible types. Can a similar result be true for solenoids which occur as minimal sets of foliations?

**QUESTION 6.2.** *Let  $r \geq 1$ , and  $\mathcal{F}$  denote the product foliation of  $M$ . Does the generic  $C^r$ -perturbation of  $\mathcal{F}$  contain a minimal set homeomorphic to a solenoid?*

If the answer to Question 6.2 is yes, one is naturally led to the following question.

**QUESTION 6.3.** *Let  $r \geq 1$ , and  $\mathcal{F}$  denote the product foliation of  $M$ . Does the generic  $C^r$ -perturbation of  $\mathcal{F}$  contain a countably infinite family of minimal sets, each of which is homeomorphic to a solenoid but no two of which are homeomorphic? Must there exist a dense such family? An uncountable such family?*

In our construction, the bonding maps were algebraically defined. In general, there will be a much wider class of covering maps possible for solenoids based on a given manifold  $L_0$ .

**QUESTION 6.4.** *Let  $r \geq 1$ , and  $\mathcal{F}$  denote the product foliation of  $M$ . There exist  $C^r$ -foliations of  $M$  which are  $C^r$ -close to  $\mathcal{F}$  containing a minimal set homeomorphic to a solenoid based on  $L_0$ . Does the requirement that the resulting foliation of  $M$  be  $C^r$  for  $r \geq 1$  eliminate homeomorphism classes of solenoids based on  $L_0$  that can occur in this way?*

For the following question, we refer the reader to the paper [3] where the notions are explained, and it is expected that such generalized solenoidal minimal sets must arise.

**QUESTION 6.5.** *Suppose that  $\pi_1(L_0, x_0) = \Gamma$  is a nilpotent group, and  $\mathcal{S}$  is a generalized solenoid based on  $L_0$  for which the leaves of  $\mathcal{F}_{\mathcal{S}}$  are simply connected. When does there exist a  $C^r$ -foliation  $\mathcal{F}$  of a compact manifold  $N$  with a minimal set homeomorphic to  $\mathcal{S}$ ?*

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